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Diffusion on a Random Comb: Distribution Function of the Survival Probability

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We investigate the distribution function $Q(P)$ describing the survival probability on a comb consisting of a backbone with lateral, randomly disconnected infinite branches. Two different regimes are analyzed in some detail: (i) at short times, $Q(P)$ is shown to have a self-similar structure (devil's staircase); (ii) at large times, this function becomes smooth and tends toward a rather well-defined unit step function. The disorder-averaged survival probability $\langle p_0(t)\rangle$ is expected to decrease as $t^{-3/4}$ at large times, whereas the relative fluctuations of the sample-dependent $p_0(t)$ display a very slow decay in time, going to zero like $t^{-1/8}$.

KEY WORDS: Fluctuation phenomena; random processes; Brownian motion.

1. INTRODUCTION

We consider the Brownian motion of a particle (walker) on a random comblike structure defined as follow.[1] At each point of a semi-infinite lattice (backbone) is attached a side branch (tooth) having a random length; two given teeth are only connected via the backbone. As a consequence, going back to the backbone is a necessary step to proceed on the latter.

Such a problem was analyzed in ref. 2 in the case of annealed disorder, i.e., when the length of a branch attached to a given site of the backbone can vary from one visit to the other. Otherwise stated, this length changes in time on a time scale comparable to or shorter than the relevant time between two visits at a given site. It was shown that when the lengths L_n are distributed according a broad law $\rho(L)$ with infinite first moment, anomalous diffusion occurs and the exponent for the mean-squared

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displacement was found. These results were obtained from two different mean-field schemes, which, generally speaking, are believed to provide correct results in the presence of annealed disorder.

We are here interested in the opposite case of *quenched* (static) disorder, i.e., when the sample is chosen once for all (the choice of L_n at any site n being assumed statistically independent of the choice at any other site $n' \neq n$). This quenched disorder generates specific time correlations since, at each visit of a given site of the backbone, the particle faces exactly the same situation. In a previous paper[3] this problem was studied with various distributions of lengths $\rho(L)$, by performing systematic expansions of infinite continued fractions related to various quantities of physical interest. It was shown that, for any $\rho(L)$, broad or narrow, the dominant term of any such expansion coincides with what can be simply deduced from a mean-field approximation, well and independently defined. It can thus be said that, for this problem, this mean-field procedure actually yields the correct asymptotics, as far as only the dominant terms are considered.

These conclusions are surprising, in view of the *low* dimensionality of the system (although it is not so easy to define it precisely, between $d=1$ and $d=2$) and of the *quenched* nature of the disorder; both these features lead one to suspect that the fluctuations play an important role and, as a consequence, that a mean-field treatment should fail, at least when the relative weight of *infinite* branches becomes important as is the case with a broad law. Generally speaking, a mean-field treatment driven with the proper variable is very often correct in normal dynamical regimes and, if so, naturally gives the dominant self-averaging terms. As an example, for a random-random walk on a disordered 1D lattice when quenched random transfer rates W_{nm}, the transport coefficients in the normal regime are indeed correctly given by the *inverse* moments $\langle (1/W_{nm})^k \rangle$; anomalous behavior precisely arises when the latter are infinite. Clearly, in such a case, mean-field treatments cannot properly describe the dynamics.

The study undertaken in ref. 3 eventually only considered the asymptotically dominant terms, which turned out to be self-averaging; we are here mainly interested in analyzing the time *decay* of the neglected subdominant terms, which are fluctuating from one sample to another. Clearly enough, according to the nature of the time decay of these fluctuations (slow or fast), the physical relevance of the mean-field results can or cannot be affected. It will turn out that, at least for the survival probability, this decay is very slow in time, whereas the expectation value indeed coincides with the mean-field result.

The orderered comb (all the teeth have the same length L, also called pure system) is a straightforward problem, which can readily be solved by the use of generating functions; the basic results are as follows. For finite L, the dynamics along the backbone is exactly the same as for a semi-infinite lattice, except for a rescaling in time $[t \to t/(L+1)]$: this plain slowing down just reflects the time spent by the particle in the branches. For instance, the probability to be at the starting point at time t decays as $t^{-1/2}$ at large times, as in a pure 1D lattice. It may be said that the probability fluid eventually fills up the visited side branches making then ineffective. On the contrary, when L is infinite, as can be anticipated in view of the results of the previous case, a new dynamical regime arises, which is anomalous since the diffusion process is characterized by nonstandard exponents;[4] as an example, the same probability now decreases as $t^{-3/4}$, faster than when L is finite, for obvious physical reasons. In this case, the saturation in probability never occurs.

The preceding reminders serve as a guide for the disordered case. Indeed, they allow us to guess that when the disorder average of the lengths $\langle L \rangle$ is finite (in particular when the latter are distributed according to an exponential law), a normal regime will occur, which is a plain generalization of what happens in the pure system. Indeed, it turns out[3] that, once the disorder has been properly sampled, the time is again simply rescaled, the scaling factor being now $1/(\langle L \rangle + 1)$. On the contrary, when the lengths of the branches are distributed according to a broad law giving a high weight to the "infinite" ones and yielding an infinite mean length, the branches cannot be saturated and some kind of anomalous regime is to be expected.

Although the previously developed mean-field approximation[3] indeed produces these two classes of asymptotic regimes, the question of the relevance of fluctuations is still open. One way to know more about this is to investigate the distribution functions describing the statistical properties of the fluctuatng terms. We here present a rather detailed study of the distribution function for the survival probability, allowing us to find the time dependence of these terms. In addition, this analysis provides an explicit example of a distribution function having a nonstandard behavior [in one regime, this function has a self-similar structure (devil's staircase), in the other one, it is continuous but nondifferentiable (singular component)]; it allows us to analyze the decay in time of the non-self-averaging terms which represent the first correction to the mean-field behavior.

The salient relevant features of disorder are here depicted by assuming a *binary* disorder: the random variable L can be equal either to 0 (with probability p) or to $+\infty$ (with probability q). Figure 1 provides a schematic view of such a binary disordered comb. Alternatively, provided that the starting point is somewhere on the backbone, the corresponding geometrical structure can be viewed as a comb with all its branches infinite, in which the first link between the backbone and the branches is randomly cut from site to site.

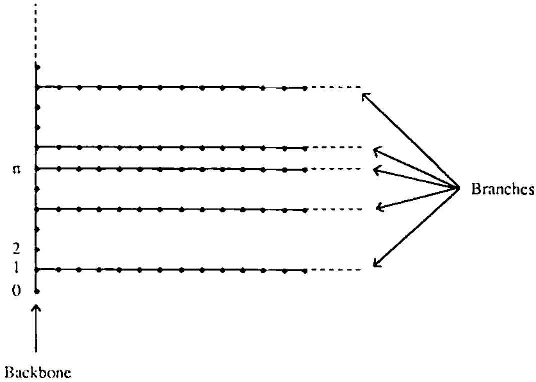

Fig. 1. Schematic drawing of a binary disordered comb with a semi-infinite backbone; the sites of the latter are numbered 0, 1, 2,.... Each branch either is absent or has an infinite length.

We here focus on the survival probability $p_0(t)$ for the particle to be still, at time t, at its starting point. Any conclusion for $p_0(t)$, which is a *local* quantity, cannot be blindly used to predict, e.g., the dynamics of the mean coordinate or its mean-squared dispersion (for instance, the expectation value $\langle p_0 \rangle$ does not allow us by itself to ascertain the existence of dynamical phases). Yet, p_0 usually plays such a central role that a new analysis of it is worthy and can shed light on some aspects of the problem. The method used here is completely different from that of our former paper[3] and relies on the study of the distribution function for the Laplace transform of $p_0(t)$; this yields an independent calculation for the asymptotics in time of its disorder average and, moreover, gives access to a study of fluctuations not undertaken in ref. 3.

2. BASIC EQUATIONS

In the overdamped limit, i.e., when inertial effects can be neglected, the motion of the Brownian particle may be described by a standard master equation with nearest-neighbor jumps, giving the time evolution of the probability to be at any given site of the whole geometrical structure; we call W (resp. w) the transfer rate along the semi-infinite backbone (resp. along any given tooth). After elimination of the motion in the branches, one gets a modified master equation involving the probabilities $p_n(t)$ to be at site n of the backbone ($n = 0, 1, 2, ...$), given that the particle is certainly at site $n = 0$ at $t = 0$ [$p_n(t = 0) = \delta_{n0}$]. $p_0(t)$ is the so-called survival probability. After performing a Laplace transformation $\{p_n(t) \to P_n(z) \equiv \mathscr{L}[p_n]\}$, we find for this modified master equation

$$x_n(z)\, P_n(z) = \frac{1}{W}\, \delta_{n,0} - (2 - \delta_{n,0})\, P_n(z) + (1 - \delta_{n,0})\, P_{n-1}(z) + P_{n+1}(z) \qquad (n \geqslant 0) \tag{2.1}$$

The quenched disorder is contained in the random function $x_n(z)$, which embodies the excursions in the branches; x_n can assume either of the two expressions $(p + q = 1,\ Z = z/W)$

$$x_n(z) = Z \equiv x_A(z), \qquad \text{probability } p \quad \text{(no branch attached to site } n\text{)}$$

$$x_n(z) = \frac{Z}{1 - e^{-\alpha}} \equiv x_B(z), \qquad \text{probability } q \text{ (branch of infinite length attached to site } n\text{)} \tag{2.2}$$

The x_n are statistically independent (clearly $0 < x_A < x_B$ for z a real, positive number). In the above equation, α is defined as

$$\operatorname{ch} \alpha = 1 + \frac{z}{2w}, \qquad \alpha \geqslant 0 \tag{2.3}$$

When Eq. (2.1) is Laplace-inverted, the left-hand side becomes a convolution involving a memory kernel with a long-time tail; as already noted in ref. 3, this prevents one from making a Markovian approximation. This fact clearly implies the strongly disordered situation here considered. x_n reduces to z/W when $w = 0$, in which case the infinite branches are inactive; in this latter case, and also obviously when $q = 0$, the dynamics is the same as for a semi-infinite lattice $[p_0(t) \sim t^{-1/2}$ at large times].

In the presence of disorder, each probability P_n becomes itself a random function, characterizable by its distribution function. In the following, we will only consider $P_0(z)$ and its distribution function Q. Namely, we define

$$\operatorname{Prob}[W P_0(z) < P] = Q(P) \tag{2.4}$$

where z is a real, positive number (so that $P_0 > 0$). $Q(P)$ is a nondecreasing positive function, vanishing for $P < 0$, and parametrized by the Laplace variable z; when the latter decreases, $Q(P)$ will be shown to change gradually, going from a devil's staircase for large z (small times) to a continuous function for small z (large times).

In order to derive the equation for $Q(P)$, let us consider the recursion relation for the survival probabilities associated to two distinct combs having respectively N and $N+1$ sites on their backbones; the N-site comb is built with the random quantities $x_1, x_2,..., x_N$; the $(N+1)$-site one starts with x_0, followed by the same sequence $x_1, x_2,..., x_N$. It is easy to show that the survival probabilities $P_0^{(N+1)}$ and $P_1^{(N)}$ satisfy the relation

$$WP_0^{(N+1)}(z; x_0, x_1, x_2,..., x_N) = \cfrac{1}{x_0 + 1 - \cfrac{1}{1 + \cfrac{1}{WP_1^{(N)}(z; x_1, x_2,..., x_N)}}} \qquad (N \geqslant 1) \tag{2.5}$$

Each of these two P's has its own distribution function, Q_N and Q_{N+1}. In Eq. (2.5), the random variable x_0 is statistically independent of all the other x_n, $1 \leqslant n \leqslant N$. Using this fact and following the method explained, e.g., in ref. 5, one finds that these functions are linked by the following relation (details are given in Appendix A):

$$Q_{N+1}(P) = p\theta\left(P - \frac{1}{x_A}\right) + q\theta\left(P - \frac{1}{x_B}\right) + pQ_N[\phi_A(P)] + qQ_N[\phi_B(P)] \qquad (P > 0) \tag{2.6}$$

θ is the unit step function [$\phi(x) = 0$ if $x < 0$, $= 1$ if $x > 0$]. The ϕ's are two noncommuting Möbius transformations defined in terms of the x_I ($I = A, B$):

$$\phi_I(P) = \frac{(x_I + 1)P - 1}{1 - x_I P} \equiv \Phi(P; x_I) \tag{2.7}$$

In the following, P_A^* and P_B^* will denote the positive fixed points of the ϕ's:

$$\phi_I(P_I^*) = P_I^* > 0, \qquad P_I^* = \frac{1}{2}\left[\left(1 + \frac{4}{x_I}\right)^{1/2} - 1\right],$$

$$\frac{1}{x_I + 1} < P_I^* < \frac{1}{x_I} \qquad (I = A, B)$$

Equation (2.6) allows us to define a recursive scheme to obtain Q_N for any N, given that Q_1 is obviously[3] equal to

$$Q_1(P) = p\theta(P - x_A^{-1}) + q\theta(P - x_B^{-1}) \tag{2.8}$$

[3] For $N = 1$ (single-site backbone), Eq. (2.1) reduces to $x_0 P_0(z) = 1/W$.

It can be proved[6] that, when $N \to +\infty$, the sequence $\{Q_N\}$ does have a limit, simply denoted as $Q(P)$, and this is the distribution function in which we are interested. This function is the fixed point (in function space) of the *functional* equation

$$Q(P) = p\theta\left(P - \frac{1}{x_A}\right) + q\theta\left(P - \frac{1}{x_B}\right) + pQ[\phi_A(P)] + qQ[\phi_B(P)] \qquad (2.9)$$

or, equivalently (the brackets $\langle \cdots \rangle$ denote averaging over the two values of x),

$$Q(P) = \left\langle \theta\left(P - \frac{1}{x}\right) + Q[\Phi(P; x)] \right\rangle \qquad (2.10)$$

$Q(P)$ displays rather unusual properties (and indeed an infinite number of singularities), as a consequence of (2.9).

Before going on, three comments are in order:

(i) For any N, $Q_N(P)$ possesses many steps (2^N), due to the presence of the θ functions; these singularities originate from discretization of space. Indeed, on a more basic level, the description of Brownian motion first involves a Fokker–Planck equation for a function $P(x, t)$ giving the probability of presence in continuous space. As ordinarily assumed in the overdamped limit, the relevant features of this equation can be captured by a master equation on a lattice with next-nearest couplings, as was done here. The functional equation (2.9) results from such a description.

Clearly enough, the introduction of second-next-nearest couplings would modify the recursion (2.5), by linking more that two survival probabilities; this would change the basic equation (2.9). One can figure out that, loosely speaking, each given step of the previous $Q(P)$ would split into several ones, tending to "regularize" each singularity. With this in mind, one can say that the incorporation of long-range couplings would eventually generate a smooth function $\bar{Q}(P)$, which can be viewed as some kind of coarse-grained average (envelope) of the initial $Q(P)$. Note that the limit $N \to +\infty$ introduces by itself a new singular feature, in the sense that the number of elementary steps is now infinite.

(ii) For $p = 1$ (resp. $q = 1$), Eq. (2.9) with the proper boundary conditions produces the expected solution $Q(P) = \theta(P - P_A^*)$ [resp. $Q(P) = \theta(P - P_B^*)$].

(iii) The mean-field approach in ref. 3 amounts to replacing Eq. (2.10) by

$$Q(P) = Q[\Phi(P; \langle x \rangle], \qquad P_B^* \leqslant P \leqslant P_A^*$$

This equation has the solution

$$Q^{mf}(P) = \theta(P - P^{mf}), \qquad P^{mf} = \frac{1}{2}\left[\left(1 + \frac{4}{\langle x \rangle}\right)^{1/2} - 1\right] \tag{2.11}$$

The small-Z expansion performed in ref. 3 shows that the dominant term in $P_0(z)$ is devoid of disorder fluctuations (self-averaging term) and is indeed given by P^{mf}/W. By using Eqs. (2.2), one readily obtains

$$Z \ll 1 : P^{mf} \approx \frac{1}{(qx_B)^{1/2}} \approx \frac{1}{\sqrt{q}} (rZ)^{-1/4} \approx \frac{1}{\sqrt{q}} P_B^* \tag{2.12}$$

So, the dominant term for P_0 behaves as $Z^{-1/4}$ at small Z.

Let us now come back to Eq. (2.9). We first show that Q goes from zero to one on a *bounded* interval. It is seen from the recursion relation (2.5) that P_0 is certainly greater than $1/(x_B + 1)$ and certainly smaller than $1/x_A$. As a consequence, we have

$$Q\left(P < \frac{1}{x_B + 1}\right) = 0, \qquad Q\left(P > \frac{1}{x_A}\right) = 1 \tag{2.13}$$

Now, let us define a "trajectory" by successive applications of the transformation ϕ_A, $P_{j+1} = \phi_A(P_j)$, $j \geqslant 1$, starting from any point P_1 to the right of P_A^*. For all the points in the interval $P > P_A^*$, Eq. (2.9) reduces to

$$Q(P) = q + pQ[\phi_A(P)] \Leftrightarrow Q(P_j) = q + pQ(P_{j+1}) \tag{2.14}$$

since then $\phi_B(P)$ is negative, implying that $Q(\phi_B(P)] = 0$ [see Eq. (2.13)]. Now, due to the instability of the fixed point P_A^*, sooner or later, one eventually obtains a P_J located to the right of $1/x_A$, for which, by Eq. (2.13), $Q = 1$; then, due to Eq. (2.14), $Q(P_{J-1}) = q + p = 1$, and so on. Thus, by tracing back the trajectory and by successive use of the set of equations (2.14), one concludes that $Q(P) = 1$ for any $P > P_A^*$. The same type of argument can be used to prove that $Q(P) = 0$ if $P < P_B^*$ (P_B^* is also unstable). We then have

$$Q(P < P_B^*) = 0, \qquad Q(P > P_A^*) = 1 \tag{2.15}$$

This allows us to concentrate on the basic equation for Q written in the relevant interval:

$$P_B^* \leqslant P \leqslant P_A^* : Q(P) = q\theta\left(P - \frac{1}{x_B}\right) + pQ[\phi_A(P)] + qQ[\phi_B(P)] \tag{2.16}$$

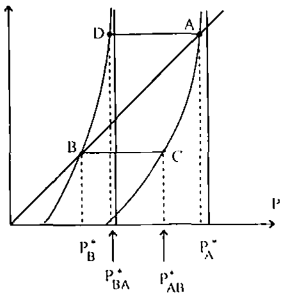

Fig. 2. Relevant parts of the curves representing the functions ϕ_A and ϕ_B defined in Eq. (2.7).

together with the boundary conditions (2.15). In the following, the "machinery" contained in Eq. (2.16) will be frequently interpreted as corresponding to the "time" evolution of a dynamical system. Figure 2 gives a schematic drawing of the relevant parts of the graphs of the ϕ's involved in Eq. (2.16). As should now be clear, the cycle $ACBD$ is in fact the basic geometrical element of the dynamic mapping.

The solution of Eq. (2.16) is $Q(P) = \mathrm{Lim}_{N \to +\infty} Q_N(P)$; $Q_N(P)$ has the expression

$$Q_N(P) = \sum_{n=0}^{N} p^n q^{N-n} \sum_{\{I\}} \theta(P - P_{\{I\}}) \qquad (N \geqslant 1) \tag{2.17}$$

$\{I\}$ denotes a configuration with A (resp. B) occurring n (resp. $N-n$) times and where $P_{\{I\}}$ is defined as

$$P_{\{I\}} = \psi_{I_1} \circ \psi_{I_2} \circ \cdots \circ \psi_{I_N}(+\infty), \qquad I_k = A \text{ or } B, \quad \psi_I = \phi_I^{-1}$$

This formal expression shows that Q_N increases from 0 to 1 by a succession of 2^N steps.

Little is known about functional equations such as Eq. (2.16), although they appear in various fields (for example, in the study of branching processes; see, e.g., ref. 7). Luck[5] explains in detail how any singularity occurring at some P_s propagates though the whole interval (here $[P_B^*, P_A^*]$) by successive applications of products $\phi_{I_1} \circ \phi_{I_2} \circ \cdots \phi_{I_N} \circ \cdots$. Here, for the purpose of further reference (Section 4.2), we will write down the scaling exponents near P_B^* and P_A^*. Let us introduce the notation

$$P_{BA}^* = \phi_B^{-1}(P_A^*), \qquad P_{AB}^* = \phi_A^{-1}(P_B^*)$$

In the interval $[P_B^*, \mathrm{Inf}\{x_B^{-1}, P_{BA}^*\}]$, Eq. (2.16) reduces to

$$Q(P) = qQ[\phi_B(P)] \tag{2.18a}$$

The solution of such an equation, constrained to vanish for $P < P_B^*$, has the form[5]

$$Q(P) = (P - P_B^*)^{\lambda_B} \Pi_B(X_B) \tag{2.18b}$$

where Π_B is a periodic function of period 1 and

$$\lambda_B = \frac{-\ln q}{\ln[\phi'_B(P_B^*)]}, \qquad X_B = \frac{\ln(P - P_B^*)}{\ln[\phi'_B(P_B^*)]} \tag{2.19}$$

For the same reasons, the solution equal to 1 for $P > P_A^*$ can be written, in the interval $[\mathrm{Sup}\{(1/x_A + 1), P_{AB}^*\}, P_A^*]$, as

$$Q(P) = 1 - (P_A^* - P)^{\lambda_A} \Pi_A(X_A) \tag{2.20}$$

where Π_A is again a function of period 1 and

$$\lambda_A = \frac{-\ln p}{\ln[\phi'_A(P_A^*)]}, \qquad X_A = \frac{\ln(P_A^* - P)}{\ln[\phi'_A(P_A^*)]} \tag{2.21}$$

The functions Π_I are the so-called periodic amplitudes and cannot be obtained by a local analysis; the scaling factor characterized by the exponent λ_I essentially provides some kind of smooth envelope $\tilde{Q}(P)$, as can be intuitively ascertained from the simple example treated in Appendix B.

It is useful to write down the asymptotic Z dependence of the exponents in the large-time limit ($Z \ll 1$). From the above formula, one finds

$$\lambda_A \approx \frac{-\ln p}{2Z^{1/2}}, \qquad \lambda_B \approx \frac{-\ln q}{2(rZ)^{1/4}} \tag{2.22}$$

Thus, as Z decreases, the envelope $\tilde{Q}(P)$ scales at the endpoints with larger and larger positive exponents and it can be anticipated that, for $Z \ll 1$, $Q(P)$ will display a rather sharp variation, expressing the decrease of fluctuations as time goes on.

Indeed, it is even worthy to go one step further and build a semiquantitative picture for $\tilde{Q}(P)$, just to see the consequences of the singular Z dependence of the exponents λ_I on the asymptotics in time of average values. Let us consider the following *model* function:

$$R(P) = C \int_{P_B^*}^{P} dP' \, (P' - P_B^*)^{\lambda_B - 1} \, (P_A^* - P')^{\lambda_A - 1} \tag{2.23}$$

where C is a normalizing factor. It may be said that dR/dP is the simplest expression which mimics the density associated to the envelope of Q, sharing with it the same exponents at endpoints. Now, R can be used to calculate a naive representation of the two averages $\langle P \rangle$ and $\sigma_P = (\langle P^2 \rangle - \langle P \rangle^2)^{1/2}$. Taking Eq. (2.22) into account, we find

$$\langle P \rangle \approx \left(1 + \frac{\ln q}{\ln p}\right)(rZ)^{-1/4}, \qquad \sigma_P \approx \frac{(2 \ln q)^{1/2}}{\ln p}(rZ)^{-1/8} \tag{2.24}$$

so that, in this crude picture, the relative fluctuation $\sigma_P / \langle P \rangle$ scales like $Z^{1/8}$. It will be seen below (Section 4) that this exponent is indeed the correct one. So, it is seen that the average values will follow power laws in time with exponents directly connected to the Lifschitz tails of the P distribution function.

It turns out that $Q(P)$ displays two quite different shapes in the two cases $Z \approx 1$ and $Z \ll 1$ relevant for small- and large-time dynamics, respectively; for the sake of clarity, they will be successively investigated. For a reason which will soon be clear, the equality defining the frontier between these two cases is $P^*_{BA} = P^*_{AB}$. This defines a unique real number Z_0; Fig. 2 precisely shows the case where $P^*_{BA} < P^*_{AB}$, i.e., $Z > Z_0$ (small times, point D to the left of point C). As will be seen, the function $Q(P)$ undergoes a transition for $Z = Z_0$: for $Z > Z_0$, it exhibits horizontal plateaus and a self-similar structure (devil's staircase). For $Z < Z_0$, $Q(P)$ becomes is an ever-increasing function, although nondifferentiable. Although we are mainly interested in the large-time regime, the detailed analysis of $Q(P)$ for $Z \sim 1$ is worthy and is the subject of the next section.

3. ANALYSIS AT SMALL TIMES

Let us consider the case $Z > Z_0$, which, loosely speaking, essentially describes the dynamics at short times. Despite this, the distribution function $Q(P)$ already possesses a complex structure due to the fact that, even at such times (even when the particle has just started to move and does not yet have an overall view of the disorder), *all* times are indeed participating in the Laplace transformation giving $P_0(z)$. This entails that, even at small times, the complexity due to the surrounding disorder already shows up.

It is easy to build a geometrical procedure allowing one to draw the graph of $Q(P)$. First we note the existence of a central plateau at a height equal to $q = 1 - p$, extending on the interval $K_0 = [P^*_{BA}, P^*_{AB}]$. Indeed, in this interval, $Q[\phi_A(P)]$ vanishes and the sum of the two other terms in the RHS of Eq. (2.16) is everywhere equal to q, so that

$$Q(P \in K_0) = q = 1 - p$$

This knowledge serves as a seed to generate the graph in an infinite sequence of nonoverlapping intervals which eventually cover the whole interval $[P_B^*, P_A^*]$. For instance, we can now find Q in the interval K_{+1} obtained by "imaging" K_0 with ψ_A; by noting $K_{+1} = \psi_A(K_0)$ and using Eq. (2.13a), one can write

$$Q(P \in K_{+1}) = q + pQ(P \in K_0) = q + pq = 1 - p^2$$

In the same way, we can define $K_{-1} = \psi_B(K_0)$ and write

$$Q(P \in K_{-1}) = qQ(P \in K_0) = q^2$$

and the same process can be repeated with $K_{\pm 1}$, and so on. We thus obtain the graph of $Q(P)$ as consisting of disconnected horizontal lines at heights q^{n+1} (in the intervals K_{-n}, $n \geqslant 0$) and $1 - p^{m+1}$ (in the intervals K_{+m}, $m \geqslant 0$); this is schematically depicted in Fig. 3: at this early stage, the graph is far from being complete. The remaining holes can be filld by other suitably chosen sequences; for instance, applying ψ_B to K_{+1}, one finds an interval located within the hole between K_{+1} and K_0. The self-similar structure of $Q(P)$ is obvious [as an example: the increase of Q between K_0 and K_{+1} (between the points α and β) is the image by ϕ_A of the increase between P_B^* and P_{BA}^*]. One can also find Q at infinitely many special points, the positive fixed points of any product of the ϕ_I; for instance, at the positive fixed point of $\phi_A \circ \phi_B$, Q is equal to $q^2/(1 - pq)$.

The above considerations allow us to devise a recursive scheme giving as many values of $Q(P)$ as desired. Consider that we want to know Q for some arbitrary value P_0 outside K_0. Take this P_0 as the starting point of a trajectory defined by the relations

$$P_{j+1} = \phi_B(P_j) \qquad \text{iff} \quad P_B^* < P_j < P_{BA}^* \tag{3.1a}$$

$$P_{j+1} = \phi_A(P_j) \qquad \text{iff} \quad P_{AB}^* < P_j < P_A^* \tag{3.1b}$$

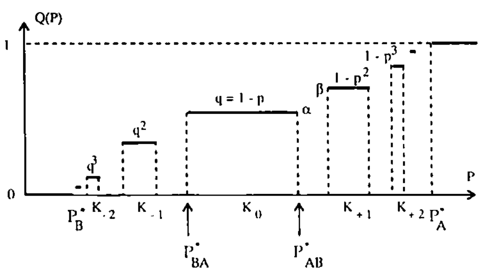

Fig. 3. Schematic view of the graph of Q at its early stage of construction (see Section 3).

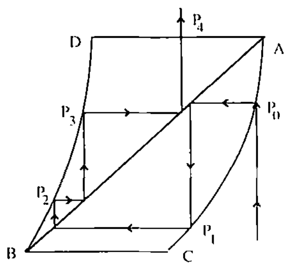

Fig. 4. Illustration of the recursive procedure allowing us to find $Q(P)$ in the small-time regime [see Eqs. (3.1)]. For simplicity, P_j denotes a point of the trajectory as well as its abscissa.

and let us stop this trajectory at the *first* iterate P_J which belongs to K_0; P_J is obtained after a finite[4] number of iterations, since the fixed points A and B are unstable, so that the trajectory bounces off the two curves ϕ_A and ϕ_B by staying within the cycle $ACBD$; it can only escape through the "doors" AD and BC. At point P_J, we know that $Q = q$.

To the above relations defining the trajectory, we associate the corresponding Eq. (2.16), which precisely reads

$$Q(P_j) = qQ(P_{j+1}) \tag{3.2a}$$

$$Q(P_j) = q + pQ(P_{j+1}) \tag{3.2b}$$

As soon as P_J is found, the RHS of Eqs. (3.2) is known, so that, by tracing back the trajectory [i.e., by inverting the chain of Eqs. (3.2)], we can successively obtain $Q(P_j)$ at all those points involved in the trajectory. For the special trajectory shown on Fig. 4 one has

$$Q(P_0) = q + pQ(P_1), \qquad Q(P_1) = q + pQ(P_2)$$
$$Q(P_2) = qQ(P_3), \qquad Q(P_3) = qQ(P_4) = q^2$$

so that

$$Q(P_2) = q^3, \qquad Q(P_1) = q + pq^3, \qquad Q(P_0) = q + p(q + pq^3)$$

It is seen that the crucial point here is that the trajectory can indeed by inverted, which results from the fact that for any P_j only *one* of the two ϕ's appearing in the RHS of Eq. (2.16) is effectively acting. In other words, each P_j has just one "son" P_{j+1}; the trajectory does not display branching

[4] This is true provided that P_0 is not exactly equal to a fixed point of any product of ϕ's; due to the finite representation of numbers in the computer, this has no practical consequences.

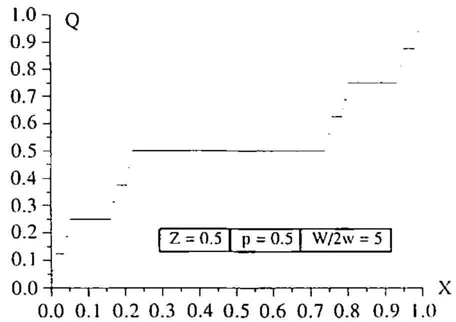

Fig. 5. Distribution function Q in the small-time regime; here, X denotes the reduced variable $(P - P_B^*)/(P_A^* - P_B^*)$. Q takes on the constant value $q = 1 - p = 0.5$ on the interval denoted as K_0 in the main text.

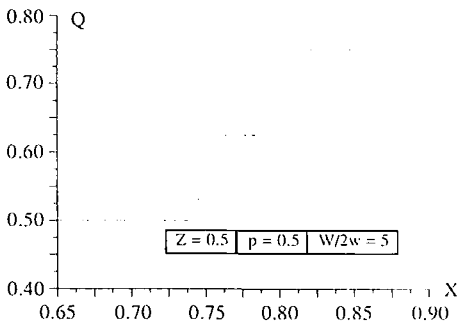

Fig. 6. Enlargement of the curve given in Fig. 5.

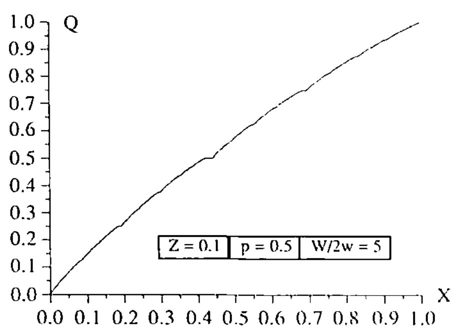

Fig. 7. Distribution function Q for times greater than in Figs. 5 and 6. X still denotes the reduced variable $(P - P_B^*)/(P_A^* - P_B^*)$.

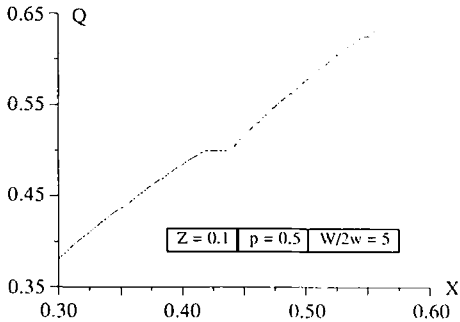

Fig. 8. Enlargement of the curve given in Fig. 7.

points, and, for this reason, can be inverted by running backward in time: this characterizes the regime $Z > Z_0$, for which K_0 exists. The whole process can be repeated for a lattice (regular or random) of initial points P_0; from a given set of starting points, one eventually obtains the values of Q at points spread over $[P_B^*, P_A^*]$ in a "chaotic" way.

The results of such a procedure are given in Figs. 5 and 6. Here Q is plotted as a function of the reduced variable X defined as

$$X = \frac{P - P_B^*}{P_A^* - P_B^*} \tag{3.3}$$

and displays the general features discussed above; Fig. 6 is just an enlargement illustrating the self-similar structure of the graph of the function Q. Figures 7 and 8 display the same kind of results for a somewhat smaller value of Z and show the tendency of K_0 to shrink when Z decreases; it is seen that the central plateau where $Q = q = 1 - p = 0.5$ is now considerably reduced, although still visible.

4. ANALYSIS AT LARGE TIMES

As Z decreases, the interval K_0 shrinks; for Z smaller than Z_0, K_0 is definitely reduced to a point (now $P_{BA}^* > P_{AB}^*$) and the problem changes in a qualitative manner. Because $P_{BA}^* > P_{AB}^*$, the basic cycle $ACBD$ is distorted (point D to the right of point C); since the prominent interval K_0 no longer exists, the above recursive scheme fails. Indeed, for any P_j in the interval $\tilde{K}_0 = [P_{AB}^*, P_{BA}^*]$, *both* terms ϕ_A and ϕ_B in the RHS of Eq. (2.16) are effectively present (and unknown). This gives rise to branched noninversible trajectories. Although it is still possible to build a recursive formal process allowing one to construct $Q(P)$ by intervals, it turns out to be of no practical utility. In the following, we will first present a purely numerical calculation of the solution of Eq. (2.16) which will serve as a reference for the sequel. Then, we will derive an approximate analytical expression which yields a good approximation for $Q(P)$.

4.1. Numerical Calculation of $Q(P)$

Equation (2.6) can be used as the basis of a simple recursive numerical calculation of $Q(P)$; it turns out that, since $1/x_A$ does not belong to the relevant interval $[P_B^*, P_A^*]$, the convergence is not so good. As a matter of fact, the same converging process can obviously be made faster simply by changing the starting point Q_1 of the whole calculation. A good choice is to begin with $Q_1 = \theta(P - P_{\text{in}})$, where P_{in} is any point in the interval $[P_B^*, P_A^*]$.[5] At the nth step, the approximate function Q_n is obtained as a linear combination of 2^n unit step functions $\theta(P - P_k)$ with weights w_k; it is then possible to compute the approximate Z-dependent average value of $P \equiv WP_0$, $\langle P \rangle^{(n)}$:

$$\langle P \rangle^{(n)} = \sum_{k=1}^{2^n} w_k P_k \tag{4.1}$$

For a given Z, this quantity gently converges with no oscillation; as a rule, the number of iterations required for having a nearly constant $\langle P \rangle^{(n)}$ [up to five figures for the average value of the reduced variable X defined in Eq. (3.3)] increases with Z^{-1} (the maximum number of iterations was $n = 14$). Obviously, the same numerical procedure can be used for any Z, in particular in the small-time regime, $Z > Z_0$.

The results of such a calculation are given in Figs. 9 and 10. In Fig. 9, we present various curves $Q(P)$ for different Z values, including the regime already known from Section 3. This set of curves displays the tendency of $Q(P)$ to become rather sharp at large times, due to the fact that, as the motion proceeds, fluctuations tend to decrease. In Fig. 10, we give the variation of $-\ln\langle P \rangle$ as a function of $-\ln Z$. It is seen that $-\ln\langle P \rangle$ converges to a straight line, establishing that

$$\langle P \rangle(Z) \to CZ^{-\beta}$$

where C is a constant prefactor and β a positive exponent. Both C and β assume the values allowing us to state that $\langle P \rangle$ and P^{mf} coincide as far as their first dominant term is considered [see Eq. (2.12)]; indeed, the

[5] It was checked that different choices of P_{in} indeed yield the same results.

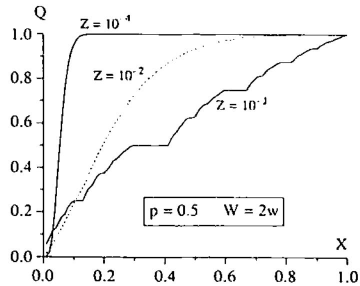

Fig. 9. Various distribution functions calculated according to the numerical scheme described in Section 4.1; the value of the parameter $W/2w$ is 1.0.

product $\langle P\rangle\langle x\rangle^{1/2}$ (dashed curve) converges rapidly to one. One can thus conclude that

$$Z \ll 1 : \langle P_0\rangle(Z) \approx \frac{1}{W}\frac{1}{\sqrt{q}}\left(\frac{W}{w}\right)^{1/4} Z^{-1/4} \tag{4.2}$$

Note that for q and/or $w=0$, the prefactor is infinite, which is consistent with the fact that, in this case, $\langle P_0\rangle(Z) \approx W^{-1}Z^{-1/2}$.

Equation (4.2) gives the dominant term of the average value of the Laplace transform of the survival probability $p_0(t)$. First, one has to be convinced that the Laplace transformation and the averaging can be performed in any order without changing the result; this can be ascertained since $Q(P)$ is actually increasing on a *bounded* interval, namely $[P_B^*, P_A^*]$; in other words, $\langle \mathscr{L}[p_0]\rangle = \mathscr{L}[\langle p_0\rangle]$ and all the analytical properties of P_0 are carried back to $\langle P_0\rangle$. Second, we know that $P_0(z)$ is analytic in the right half-plane Re $z \geqslant 0$, so that the same is true for the average $\langle P_0\rangle$; the above asymptotic expansion (4.2) can thus be analytically continued in the vicinity of the origin, outside the real axis. Using the fact that, from its very

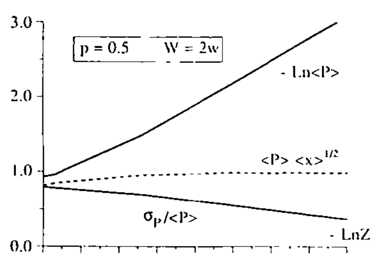

Fig. 10. Variations as a function of $-\ln Z$ of the average values deduced from the distribution function calculated according to the numerical scheme described in Section 4.1 ($p=0.5$, $W=2w$).

definition, $P_0(z)$ tends to zero for $z \to +\infty$, $\operatorname{Re} z > 0$, and further explicitely assuming that any other singularity of P_0 stands at a *finite* distance of $Z = 0$, we eventually can find the behavior of $\langle p_0 \rangle$ at large times[8]; we obtain

$$t \to +\infty : \langle p_0 \rangle(t) \approx \frac{1}{\Gamma(1/4)} \frac{1}{\sqrt{q}} \left(\frac{W}{w}\right)^{1/4} (Wt)^{-3/4} \tag{4.3}$$

Equation (4.3) shows that the average $\langle p_0(t) \rangle$ asymptotically behaves in time as $p_0(t)$ does in a pure system in which all the branches are infinite ($t^{-3/4}$). Disorder only appears through the prefactor, which embodies the probability q and the branch efficiency (w).

The simple numerical calculation described above does not allow us to obtain the *precise* Z dependence of the fluctuation of P, for the following reasons. By its definition, the dispersion $\sigma_P^2 = \langle P^2 \rangle(Z) - \langle P \rangle^2(Z)$ involves a second moment which, as a rule, enters its asymptotic regime in a slower way than a first moment does (one has to wait a longer time before the subdominant terms are negligible). This means that in order to find the exponent for σ_P, one would have to investigate much smaller Z values, for which the previous numerical procedure has such a poor convergence that an alternative method is required.

Figure 10 also shows the relative dispersion $\sigma_P/\langle P \rangle$ and already displays the slow decrease of the relative fluctuation. It is interesting to note that, on the contrary, $\langle P \rangle$ rapidly converges to $\langle x \rangle^{-1/2}$ (the mean-field value) much before than the fluctuations are substantially reduced. Thus, although $p_0(t)$ is certainly less and less fluctuating as time goes on, the decay of its fluctuations can be expected to be rather slow. For the reasons explained above, we now look for an analytical approximate expression in order to get the asymptotics of these fluctuations.

4.2 An Analytical Approximation for $Q(P)$ and Analysis of Fluctuations

Our aim is now to derive an approximate expression for $Q(P)$ valid at small Z. This must be done with great care, as can be inferred from the following remark. It can be seen that for $Z \ll 1$, the graph of ϕ_A is extremely close to the first bisector, so that it is very tempting to simply substitute $\phi_A(P)$ by P in the basic equation (2.16). We are left with

$$Q(P) \approx Q[\phi_B(P)] + \theta\left(P - \frac{1}{x_B}\right)$$

and the weight q just disappears. The solution of this equation, subjected to the boundary conditions (2.15), is clearly $\theta(P-P_B^*)$, which misses the important factor $q^{-1/2}$ (and gives $\langle P\rangle \approx P_B^*$).

From the previous numerical calculation, we see that, for $Z \ll 1$, the only relevant interval for P stands in the vicinity of P_B^*. This can also be inferred from Eq. (2.22), which shows that, since for $Z \ll 1$ the exponent λ_A diverges as $Z^{-1/2}$, $Q(P)$ is already quite close to 1 far before P reaches P_A^*. This is due to the fact that for $Z \ll 1$, ϕ_A is quasilinear and quite close to the first bisector; so, in order to go from one point in the middle of the basic interval to another one in the vicinity of P_A^*, one needs many bounces and each of them introduces a factor $p<1$. Since λ_B is also diverging (as $Z^{-1/4}$, that is, less rapidly than λ_A), all the Z-exponents for the averages are expected to be controlled by λ_B, i.e., by Lifschitz tail at the *lower* edge of the distribution.

It is thus be recognized that, as a whole, $Q(P)$ is becoming steeper and steeper, the (random) variable P being concentrated near P_B^*, which is of the order of $Z^{-1/4}$, whereas x_A and x_B are small parameters of the order of Z and $Z^{1/2}$, respectively. Precisely

$$x_B \approx (rZ)^{1/2}, \qquad P_B^* \approx \frac{1}{\sqrt{x_B}}, \qquad P_A^* \approx \frac{1}{\sqrt{x_A}} = Z^{-1/2}$$

This allows us to expand the ϕ's as

$$\phi_A(P) \approx P - 1 + O(Z^{1/2}), \qquad \phi_B(P) \approx P + (x_B P^2 - 1) + O(Z^{1/4}) \quad (4.4)$$

Now, Q approximately satisfies

$$Q(P) = pQ(P-1) + qQ(P + x_B P^2 - 1) \quad (4.5)$$

Up to this point, no drastic approximation has been actually done; the latter equation is still a functional one and its solution still retains the essential features of $Q(P)$. We now assume that the function $Q(P)$ can be approximated by a smooth differentiable function Q_{ap}, which may be seen as a coarse-grained average of Q. An equation for Q_{ap}, valid up to an unessential cutoff, can be written down simply by expanding Eq. (4.5) in a Taylor series truncated after the second term. This gives for $\rho = dQ_{\mathrm{ap}}/dP$:

$$\left[(P^2 - P_B^{*2})^2 + \frac{p}{q} P_B^{*4}\right] \frac{d\rho}{dP} + 2P_B^{*2}\left(P^2 - \frac{1}{q} P_B^{*2}\right)\rho(P) = 0 \quad (4.6)$$

At this point, several comments are in order. First, $d\rho/dP$ vanishes for $P = P_B^*/\sqrt{q}$, which precisely coincides with the result expressed by Eqs. (4.2) and (2.12); in other words, ρ has its maximum at $P = P^{\mathrm{mf}}$. Second, the crudest approximation in Eq. (4.6) should be to forget about the derivative, which produces the solution $\delta(P-P_B^*/q^{1/2})$, again consistent with Eq. (4.2); indeed, this is strictly the mean-field result, as expressed by Eqs. (2.11) and (2.12). Finally, the solution of (4.6) no longer has the specific singularities as described in Eqs. (2.18b). This distinction is hoped to be irrelevant since it implies only the close vicinity of P_B^*, in which Q is extremely small when $Z \ll 1$ $[Q \approx (P-P_B^*)^{(-\ln q)(rZ)^{-1/4}}]$. Finally, Eq. (4.6) is expected to provide an approximate $Q_{\text{ap}}(P)$ wich is better and better as Z becomes smaller and smaller.

Equation (4.6) can be readily integrated; the approximate density associated to $Q(P)$, Q'_{ap}, can be eventually expressed in terms of a complex number A and its complex conjugate A^*:

$$Q'_{\text{ap}}(P) = CA^*A, \qquad A = \left(\frac{\xi P_B^* + P}{\xi P_B^* - P}\right)^{\xi P_B^*/2}, \qquad P_B^* \approx (rZ)^{-1/4} \tag{4.7}$$

C is a normalization factor and ξ the complex number

$$\xi = \frac{1}{\sqrt{2}} \left[(q^{-1/2}+1)^{1/2} + i(q^{-1/2}-1)^{1/2}\right]$$

From this expression, we can deduce Q_{ap} by a simple numerical integration. The result of this last calculation is shown in Fig. 11 for $Z = 10^{-4}$, together with the "exact" result (dashed curve) obtained from the purely numerical scheme developed in Section 4.1 [in addition, the shown unit step function is the mean-field result (see Eq. (2.11); here, one has $\langle X \rangle \equiv Z^{1/4}(q^{-1/2}-1) \approx 0.041$]. The horizontal scale has been dilated in order to display the important part of the curves and to make their differences clearly visible. It is seen that the two curves are indeed quite close to one another (they stand closer and closer as Z becomes smaller and smaller). This agreement may be considered as establishin the correctness of the above analytical procedure on numerical grounds, since there is no obvious justification for it (although such a kind of expansion is frequently used in practice; see, e.g., ref. 9). We thus explictely claim to have *only* a numerical justification for an analytical expression in which the parameters of the model can now be varied at ease. Note the Z dependence in the exponent of the quantity A [Eq. (4.7)], which goes to infinity if w and/or if q goes to zero. The large exponent behaving as $Z^{-1/4}$ entails that Q'_{ap} does have a very rapid variation in the large-time limit; at infinite times, it becomes a δ-function.

We give in Fig. 12 the two approximate densities Q'_{ap}, plotted as functions of the reduced variable X [see Eq. (3.3)]. This figure displays

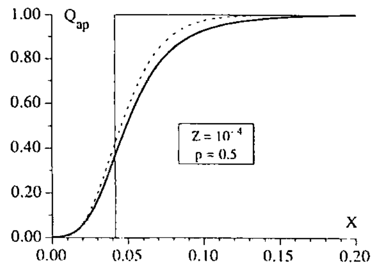

Fig. 11. Analytical approximation Q_{ap} as a function of the reduced variable X (solid curve); the dashed curve, shown for comparison, was calculated according to the numerical scheme described in Section 4.1 (here $W = w$); the thin unit step curve is the mean-field result as expressed by Eq. (2.11).

qualitatively the decrease of the average value $\langle X \rangle$ and of its fluctuation σ_X as Z decreases; it also shows that σ_X is hardly smaller than $\langle X \rangle$ even for $Z = 10^{-6}$.

Although we now know an approximate explicit expression for the density Q'_{ap}, the latter is not so easy to handle analytically. In order to readily find the Z dependence of fluctuations, we take advantage of the fact that the exponent ξP_B^* has a very large modulus, allowing us to calculate $\langle P^2 \rangle$ and $\langle P \rangle$ by the Laplace method.[11] A straightforward but somewhat tedious calculation yields the following expressions:

$$\langle P \rangle = \left(\frac{1}{\sqrt{q}} + \frac{7p}{8qP_B^*} + \cdots \right) P_B^* \tag{4.8}$$

$$\sigma_P^2 = \frac{p}{4q^{3/2}} P_B^* \tag{4.9}$$

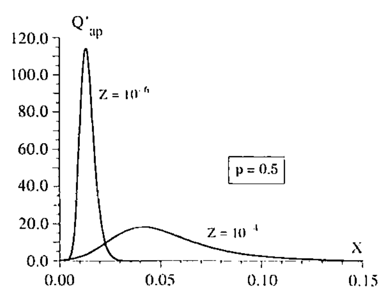

Fig. 12. Analytical approximation Q'_{ap} for $Z = 10^{-4}$ and $Z = 10^{-6}$ displaying the fact that the dispersion is hardly smaller than the average value even for very small Z.

Equation (4.8) shows that the first relative correction to the average value is $\sim 1/P_B^* \sim Z^{1/4}$. Equation (4.9) gives the interesting sought result: the mean-squared dispersion of P diverges as $Z^{-1/4}$ as Z goes to zero. From this, we can deduce the asymptotic time behavior of the autoconvolution of $p_0(t)$:

$$\int_0^{+\infty} dt' \, \langle p_0(t-t')\, p_0(t')\rangle - \langle p_0(t-t')\rangle \langle p_0(t')\rangle$$

$$\sim W^{-1} \frac{1}{\Gamma(1/4)} \frac{p}{4q^{3/2}} \left(\frac{W}{w}\right)^{1/4} (Wt)^{-3/4}$$

It results that the mean-squared dispersion for $p_0(t)$ has the asymptotic decay

$$\begin{aligned} \sigma_{p_0}^2(t) &\equiv \langle p_0(t)\, p_0(t)\rangle - \langle p_0(t)\rangle \langle p_0(t)\rangle \\ &\sim \frac{p}{4q^{3/2}} \left(\frac{W}{w}\right)^{1/4} \frac{[\Gamma(1/8)]^{-2}}{(Wt)^{7/4}} \end{aligned} \tag{4.10}$$

implying that at large times

$$\frac{\sigma_{p_0}(t)}{\langle p_0\rangle(t)} \sim \frac{p^{1/2}}{2q^{1/4}} \left(\frac{w}{W}\right)^{1/8} \frac{\Gamma(1/4)}{\Gamma(1/8)} \frac{1}{(Wt)^{1/8}} \tag{4.11}$$

We can thus *symbolically* write, for a given sample,

$$p_0(t) \sim \langle p_0(t)\rangle [1 + \eta(\{L_n\})(Wt)^{-1/8}] \tag{4.12}$$

where η denotes a noise of order t^0 depending on the given configuration of the lengths. Thus, although $p_0(t)$ is certainly less and less fluctuating as time goes on, the decay of the fluctuations turns out to be extremely slow.

5. SUMMARY AND CONCLUSIONS

In the present work we analyzed in detail the average and the fluctuations of the survival probability $p_0(t)$ in the large-time regime.

The averaged probability $\langle p_0\rangle(t)$ behaves in the asymptotic regime as already found in ref. 3 and is actually correctly given by the mean-field treatment there defined; the convergence of this average to the mean-field value is a rather rapid one. The asymptotic time dependence of $\langle p_0(t)\rangle$ is the same as for an ordered comb in which *all* the teeth have an infinite length; the disorder only enters through a prefactor diverging when the probability q of having an infinite tooth tends toward zero.

On the other hand, the behavior of sample-to-sample fluctuations is markedly different, as shown by a detailed study of the distribution function $Q(P)$ describing the random variable $P_0(z)$, the Laplace transform of $p_0(t)$. $Q(P)$ undergoes a transition for some value Z_0 of the Laplace variable, going from a devil's staircase to a smooth function. A numerical calculation of $Q(P)$ in this latter case was used for two purposes; first, it allowed us to give a first view of the slow decrease of fluctuations as the motion proceeds in time (and also the correctness of the average value $\langle P_0 \rangle$ previously obtained); second, it was used as a check for an analytical approximate expression for $Q(P)$ displaying explicitly the parameters of the model and, essentially, the singular dependence upon the Laplace variable. With this, it was eventually possible to find the precise time decay of the fluctuations of the non self-averaging first correction to the mean-field behavior; it was shown that the relative fluctuations undergo a very slow decay in time, according to a $t^{-1/8}$ law. This makes the mean-field result valid only at extremely large times, once the unavoidable fluctuations linked to quenched disorder and low dimensionality have been actually washed out. As a consequence, the mean-field result is, at intermediate times, of no physical relevance.

The discussion of the present paper focused on the survival probability $p_0(t)$, which, although usually playing a central role in this kind of problem, is obviously far from providing a complete description of the dynamics. Due to the fact that this simple quantity already shows large fluctuations with a very slow decay in time, it is tempting to conclude that, as a whole, the dynamics probably also displays physically relevant non-self-averaging features. The mean coordinate $\langle x \rangle(t)$ and the mean squared displacement $\Delta x^2(t)$ are obtained by performing summation over all the p_n; such a procedure, involving strongly fluctuating nonindependent random variables, can either enhance or reduce the overall fluctuations. Due to this uncertainty, it may be thought that the large-time behavior of $\langle x \rangle$ and of Δx^2 is still a largely open question.

APPENDIX A

Because Eq. (2.6) is one of the basic equations of the present work, we here give its full derivation. The starting point is Eq. (2.5) simply rewritten as

$$P^{(N+1)} = \frac{1}{x_0 + 1 - \dfrac{1}{1 + \dfrac{1}{P^{(N)}}}} \equiv \frac{P^{(N)} + 1}{x_0 P^{(N)} + x_0 + 1} \tag{A1}$$

where it is understood that $P^{(N)}$ is statistically independent of x_0. The distribution function $Q_{N+1}(P)$ is, by definition, the probability that $P^{(N+1)}$ is smaller than P:

$$Q_{N+1}(P) = \mathrm{Prob}[P^{(N+1)} < P] \tag{A2}$$

The event $P^{(N+1)} < P$ can be realized in two mutually exclusive ways:

$$\text{(i)} \quad P < \frac{1}{x_0}: P^{(N+1)} < P \Leftrightarrow -1 - \frac{1}{x_0} < P^{(N)} < \frac{(x_0+1)\, P^{(N)} - 1}{1 - x_0 P^{(N)}} \tag{A3}$$

or

$$\text{(ii)} \quad P > \frac{1}{x_0}: P^{(N+1)} < P \Leftrightarrow P^{(N)} < \frac{(x_0+1)\, P^{(N)} - 1}{1 - x_0 P^{(N)}}$$

$$\text{or} \qquad P^{(N)} > -1 - \frac{1}{x_0} \tag{A4}$$

For the moment, let us call f the density describing the random variable x_0; from Eqs. (A1)–(A4) and introducing the unit step function θ, we can now write

$$\begin{aligned} Q_{N+1}(P^{(N+1)}) &= \int f(x_0)\, dx_0 \left\{ \theta\left(\frac{1}{x_0} - P^{(N)}\right) \right. \\ &\quad \times \left[Q_N\left(\frac{x_0+1)\, P^{(N)} - 1}{1 - x_0 P^{(N)}}\right) - Q_N\left(-1 - \frac{1}{x_0}\right)\right] \\ &\quad \left. + \theta\left(P^{(N)} - \frac{1}{x_0}\right)\left[Q_N\left(\frac{(x_0+1)\, P^{(N)} - 1}{1 - x_0 P^{(N)}}\right) + 1 - Q_N\left(-1 - \frac{1}{x_0}\right)\right]\right\} \end{aligned}$$

Since P is certainly positive, then $Q_N(-1 - 1/x_0) = 0$; this equation can be rewritten as

$$Q_{N+1}(P^{(N+1)}) = \int f(x_0)\, dx_0 \left[Q_N\left(\frac{(x_0+1)\, P^{(N)} - 1}{1 - x_0 P^{(N)}}\right) + \theta\left(P^{(N)} - \frac{1}{x_0}\right)\right] \tag{A5}$$

Taking now into account the binary nature of the presently considered disorder, $f(x_0) = p\delta(x_0 - x_A) + q\delta(x_0 - x_B)$, we obtain Eq. (2.6) of the main text.

APPENDIX B

As an example, let us consider the simple equation

$$f(x) = qf(\alpha x)$$

where q and α are real numbers ($0 < q < 1$, $\alpha > 1$). The function[6] f is assumed to satisfy the boundary conditions

$$f(x) = 0, \qquad \forall x < 0; \qquad f(x) = C, \qquad \forall x \geqslant x_1$$

where C and x_1 are positive real numbers. The unique resulting solution can be easily guessed and is found to be

$$f(x) = Cq^{-\mathrm{Int}[\ln(x/x_1)/\ln \alpha]} \tag{B1}$$

where Int denotes the integer part. This function displays a countable number of steps between 0 and x_1 and its graph also displays a self-similar structure. On the other hand, if one looks for a solution scaling like x^λ near $x = 0$, one obtains

$$\tilde{f}(x) = Cx^{-\ln q/\ln \alpha} \tag{B2}$$

Clearly, these two expressions are different, but the expression (B2) is easily seen to simply wash out all the steps occurring in (B1) and actually produces some kind of "averaged" function (coarse-grained envelope) which is a rather good approximation of (B1) in the vicinity of $x = 0$.

On a technical level, it evidently possible to recover exactly (B1) by looking for a solution of the form x^λ. The point is that the equation giving λ is

$$1 = q\alpha^\lambda$$

and has an *infinite* number of solutions, of the form

$$\lambda = \frac{2i\pi n - \ln q}{\ln \alpha}, \qquad n \in \mathbb{Z} \tag{B3}$$

The naive scaling yielding (B2) only retained the solution $n = 0$. It is a simple exercise on Fourier series to show that, with the whole set of solutions (B3), the exact form (B1) is recovered.

[6] With another boundary condition, this equation is known as the Koenig equation;[9] the Koenig theorem then states that there exists a unique solution.

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